Mulifractal phase transitions: the origin of self-organized criticality in earthquakes

C. Hooge, S. Lovejoy, D Schertzer, S. Pecknold, J.-F. Malouin, François G Schmitt

Nonlinear Processes in Geophysics (1994) 1: 191 - 197

Nonlinear Processes in Geophysics

Multifractal phase transitions: the origin of self-organized criticality in earthquakes

C. Hooge[1], S. Lovejoy[1], D. Schertzer[2], S. Pecknold[1], J.-F. Malouin[1] and F. Schmitt[2]

[1] Department of Physics, McGill University, 3600 University Street, Montréal (Québec), H3A 2T8, Canada
[2] Laboratoire de Météorologie Dynamique (CNRS), Université Pierre et Marie Curie, 4 Place Jussieu, 75252 Paris Cedex 05, France

Received 4 February 1994 - Accepted 22 April 1994 - Communicated by S.S. Moiseev

Abstract. Fractal and occasionally multifractal behaviour has been invoked to characterize (independently of their magnitude) the spatial distribution of seismic epicenters, whereas more recently, the frequency distribution of magnitudes (irrespective of their spatial location) has been considered as a manifestation of Self-Organized Criticality (SOC). In this paper we relate these two aspects on rather general grounds, (*i.e.* in a model independent way), and further show that this involves a non-classical SOC. We consider the multifractal characteristics of the projection of the space-time seismic process onto the horizontal plane whose values are defined by the measured ground displacements, we show that it satisfies the requirements for a first order multifractal phase transition and by implication for a non-classical SOC. We emphasize the important consequences of this stochastic alternative to the classical (deterministic) SOC.

1 Introduction

One of the oldest scaling laws in geophysics is the Omori law (Omori, 1895). It describes the temporal distribution of the number of aftershocks which occur after a larger earthquake (*i.e.*, the mainshock) by a scaling relationship (power law). In the 1980's, due to the impetus of fractal geometry, scaling ideas were also applied to the spatial distribution of earthquakes. Others have shown that hypocenters and epicenters of earthquakes could be treated as geometric fractal sets whose scaling could be characterized fractal dimensions ranging between $1.1 \sim 1.6$ (Kagan and Knopoff, 1980; Sadovskiy et al., 1984; Okubo and Aki, 1987; Aviles et al., 1987; Hirata et al., 1987; Hirata, 1989). Recent multifractal analyses (Geilikman et al., 1990; Hirabayashi et al., 1992; Hirata and Imoto, 1991) of the spatial density of earthquakes have confirmed the spatial scaling and have given a more complete description.

While the Omori law coupled with the scaling of the positions (and corresponding spatial density) of the seismic events clearly show that the dynamics underlying the occurrence of seismic events is a scaling space-time process, it still provides only a very limited description. This is because the spatio-temporal position of events does not take into account their intensities which for earthquakes have long been known to vary tremendously even at a fixed location. Conversely, the other basic empirical seismological law, the Gutenberg-Richter law (Gutenberg and Richter, 1944), ignores an event's space-time location, and relates its intensity (amplitude and hence magnitude) to its probability of occurrence. It is therefore natural to combine the two types of information — *i.e.* on the one hand the space-time location of seismic events in a given area and during a given period, and on the other hand the intensity of each event — into a space-time process whose values are the intensities. In this paper, in order to have the highest possible density of events, we will pursue the slightly more modest approach of considering only the spatial projection of such a process. We will however make an important extension of previous analyses by systematically considering the different powers η of the process. One may note already that the above mentioned geometric studies of the density of epicenters corresponds to $\eta = 0$. To our knowledge there has been until now a single multifractal study for $\eta \neq 0$, using the value $\eta = 1.5$ which is an estimate of the distribution of seismic energy (Hirabayashi et al., 1992). In any event, the treatment of such generalized seismic fields takes us beyond geometric considerations on the space or time distribution of the centers to consider processes[1].

During the 1980's it became increasingly clear that whereas the general framework for scaling geometric sets

[1] In the following we will employ interchangebly the terms fields and processes to indicate space-time dependencies although the former emphasizes the spatial dependency while the latter, the temporal dependency.

Correspondence to: C. Hooge

was fractals, for scaling processes it was rather multifractals. Furthermore, it was recognized that a generic feature of the general (stochastic, canonical) multifractals was the appearance of qualitatively different weak-/strong soft/hard behaviour — initially termed hyperbolic intermittency (Schertzer and Lovejoy, 1985) — also characterized by power law probabilities (Schertzer and Lovejoy, 1987, 1992). Due to the existence of a formal analogy between multifractals and thermodynamics, qualitative changes of this sort are termed *multifractal phase transitions*; the soft/hard transition discussed here is an example of a first order, low temperature transition (Schertzer et al., 1993). Recently this combination of spatio-temporal scaling with power law probabilities has been taken as the hallmark of *Self-Organized Criticality* (SOC, Bak et al., 1987) and it has been argued that this is the result of deterministic rather than stochastic "toy" models. Several earthquake models of this sort have since been proposed (e.g. Ito and Matsuzaki, 1990). However many criticisms of this "classical" SOC scenario have been made. For example, it is not consistent with the presence of foreshocks or aftershocks (Barriere and Turcotte, 1991). This defect is fundamental since it results from the fact that classical SOC cannot deal with interacting avalanches (*i.e.* events): it requires a vanishing flux whereas stochastic SOC deals with non–zero flux and interacting avalanches (see discussion in Schertzer and Lovejoy (1994b)).

Below, using multifractal analysis techniques involving the different normalized powers η of space-time *seismic processes*, we simultaneously analyze the position and amplitude of the seismic processes (using the USGS catalogue). As mentioned earlier, we omit the time dependency proceeding to a multifractal analysis of the projection on the space of the space-time process, preserving its multifractal space-intensity properties.

We go on to show that the critical (generalized Gutenberg-Richter) exponents characterizing the multifractal phase transitions obey a relationship predicted by multifractal theory. We specifically show that the critical orders of statistical moments ($q_{D,\eta}$) of the first order multifractal phase transition of the η (normalized) power of the process, generalize the Gutenberg-Richter exponent $b(\eta) \equiv q_{D,\eta}$ (the usual Gutenberg-Richter exponent is $b = b(1)$). Indeed, the statistical moment scaling exponent of order q of the η (normalized) power of the process, $K(q,\eta)$, follows a special theoretically predicted linear form: $K(q_{D,\eta},\eta) = (q_{D,\eta}-1)D$ for $q \geq q_{D,\eta}$ where D is an empirical constant. By varying η, we are able to determine the non-linear dependence upon η of the critical order moment $q_{D,\eta}$. The value of D is shown to be independent of the parameter η. Since $q_{D,\eta}$ changes with η while D does not, D is a more fundamental constant with which to describe the earthquake process. These results show that the origin of self-organized criticality in earthquakes may be in stochastic, space-time tensorial multifractal processes.

2 Normalized Powers Of Seismic Processes

Scaling ideas have evolved rapidly since the early 1980's and many geophysical fields or processes have now been shown to be scaling, sometimes over very large ranges of space and time scales. Indeed, it has been argued for some time (e.g. Schertzer and Lovejoy (1991) and references therein) that this ubiquity is not surprising since scaling can be regarded as a symmetry principle. Viewed in this way, geophysical systems are expected to be scaling because few geophysical processes have specific mechanisms which operate at unique scales and which are strong enough to break the scaling. However, treating scale invariance as a symmetry principle does more than simply explain the presence of scaling; it gives us quite specific predictions about the overall dynamics and statistics. For example, when nonlinear dynamical processes are scale invariant, it is now becoming clear that the resulting fields are multifractals, whereas associated scale invariant geometric sets are fractals. Various theoretical properties of multifractals can then be exploited including the occurrence of rare but violent events ("hard" behavior) and the possibility of universality (*i.e.*, behaviour independent of many of the details of the process, see Schertzer and Lovejoy (1987, 1992), in earthquakes, see Hooge (1993)).

The basic seismological fields are the stress and strain tensors, and given the evidence for scaling discussed above, the natural framework is multifractal tensor processes (see Schertzer and Lovejoy (1994a) for the generalizations of multifractals beyond positive scalars using Lie cascades). However, the stress and strain tensors are generally not directly observable; seismic observations are based on the ground displacements of each event. This data is then used (via inversion techniques) to determine the position of the hypocenter, the origin time, and seismic moment tensor. Consider a seismic zone size L. The natural way to create a multifractal field is therefore to use a grid size $l < L$ and sum the amplitudes over all the events that occur within each grid element. Since by itself the sum of maximum ground motion amplitudes A over a grid has no obvious physical significance, we are thus lead to define the various (normalized) powers of *seismic fields*, indexed by the parameter η:

$$S_{\eta,\lambda} = \frac{\int_{B_\lambda} (A_\Lambda)^\eta d^d x}{\int_{B_\lambda} d^d x}, \tag{1}$$

where the subscript $\lambda = L/l (> 1)$ denotes the resolution of the seismic field. The subscript $\Lambda \gg \lambda > 1$ indicates the very small intrinsic resolution of the catalogue data. d denotes the dimension of the constructed seismic field (here, $d = 2$, the earth's surface), and B_λ is a grid box scale λ (size L/λ). The denominator normalizes the integrated η power of the ground displacement. When $\eta = 0$, each event is given the same weight; $S_{0,\lambda}$

will be the density of the number of events at scale λ, the statistics will be the same as in the geometric multifractals references discussed above. Since semi-empirical models of earthquake processes relate the amplitudes to moments and energies of individual events, seismic fields with specific values of η (such as $\eta = 1.5$ for seismic energy (Hirabayashi et al., 1992)) could be regarded (due to the normalization) as generalized moment or energy fields. Similarly, by studying probability distribution of $S_{\eta,\lambda}$, we will obtain a family of exponents indexed by η which are generalizations of the Gutenberg-Richter exponents (below, we show that with $\eta = 1$, the generalized (normalized) exponent equals the usual, (unnormalized) one). As we increase the parameter η we place increasing weight on the extreme events; by studying the statistical properties of the entire family of $S_{\eta,\lambda}$ as functions of resolution λ, we obtain a complete characterization of the scaling properties of the earthquake catalogue. This technique has the advantage that as η increases, it is less and less sensitive to the minimum detection of the network, unlike either the box-counting or pair correlation techniques mentioned above (Hooge, 1993). Presumably, since multifractals are generic scaling fields, if the seismic fields are multifractal then the nonlinear process which generated these fields is also multifractal. A fundamental problem in seismology will be to relate the scaling properties of the seismic fields to those of the underlying tensor fields.

One may note that for the present time we have only scalar and pointwise data. The effect of the latter may not be too severe since only some of the larger events will have rupture areas larger than our resolution ($\approx 2km$). Due to the sparseness of the network, some weaker events may be missed. However, the measuring network can be seen as another, independent, multifractal phenomenon (see Tessier et al. (1994)), and given this, will not break the scaling (although it may modify the multifractal exponents). A more significant limitation is that our scalar analysis is unable to take into account the strong anisotropy of individual events associated with fault directions; Lie analysis (Schertzer and Lovejoy, 1994a) is required to proceed with more sophisticated data including tensorial information. However, there is still a strong anisotropy of the observed scalar process; this takes us beyond self-similar processes requiring the framework of Generalized Scale Invariance (Schertzer and Lovejoy, 1985), and will be investigated in future papers.

In this study, we treat each earthquake as a point process. On the one hand this treatment is similar to previous studies of the density of events —we simply consider η not restricted to only zero —on the other hand this simplistic assumption will only affect our definition of seismic fields for the few earthquakes which have rupture areas larger than the minimum resolution of the constructed seismic field (which is typically around $2km$).

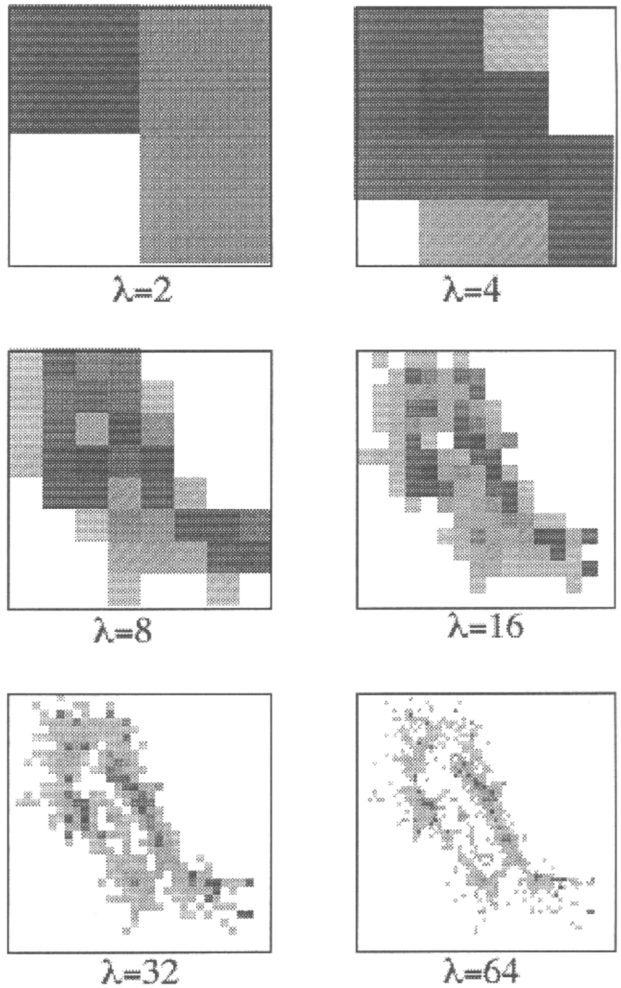

Fig. 1. The seismic field for various values of the parameter λ with $\eta = 1$. This figure shows how the picture changes with resolution (*i.e.* λ).

3 Multiscaling Properties Of Seismic Fields

To test these ideas we used data from the local earthquake catalogue of earthquakes in Central California compiled by the U. S. Geological Survey (USGS) at Menlo Park, California. This study is based on earthquakes occurring between January 1^{st}, 1980 and December 31^{st}, 1990 (approximately 4000 days). These earthquakes were situated in an area bounded by the lines of North latitude 33°30" and 43°10" and lines of West longitude by 115°00" to 128°48". There were approximately 235,000 earthquake events in this catalogue which uses information derived from the California seismic detection network which now, for example, comprises more than 300 seismological stations (there were roughly 200 stations in operation as of 1990 — see Marks and Lester (1980)). The fields for several values of λ are shown in Fig. 1 using a gray scale rendering. Figure 2 shows the same region at maximum resolution (*i.e.* $\lambda = 512$). The maximum ground motion (normalized

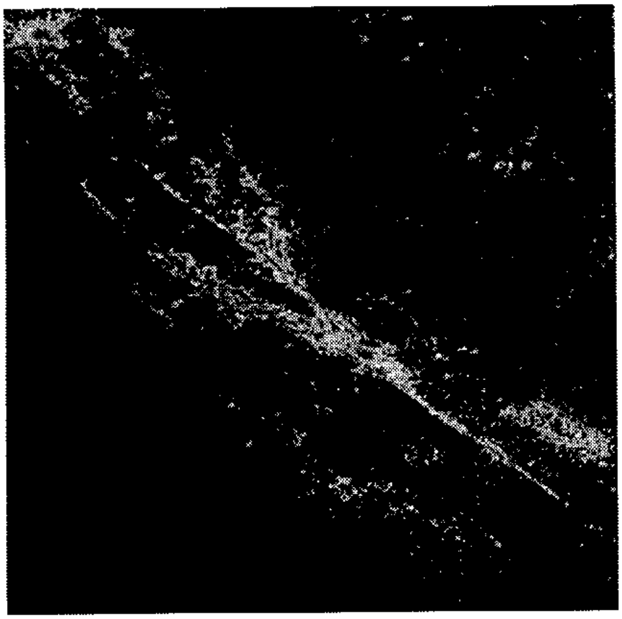

Fig. 2. The seismic field at the finest resolution (512 x 512 grid) with $\eta = 1$.

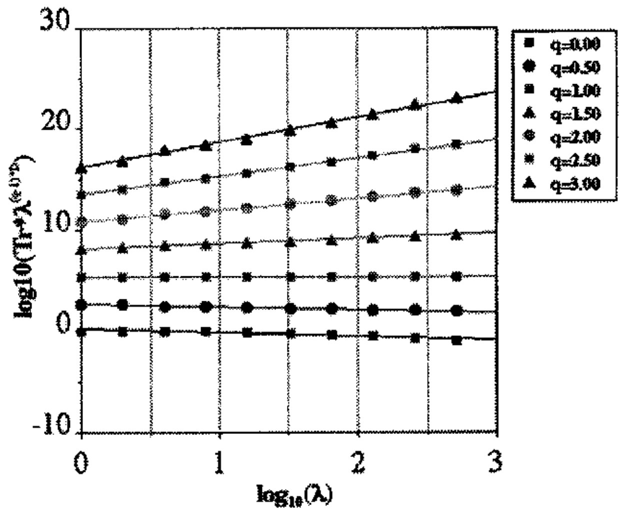

Fig. 3. Multiscaling of statistical moments for earthquakes in the period 1980–90. The range of scaling for this (and all subsequent earthquake analyses) is from $2km$ ($\log_{10}(\lambda) = 2.7$) to $1000km$ ($\log_{10}(\lambda) = 0$).

to 50 km from the epicenter) was determined from the catalog. The depth and time coordinates were ignored, that is, only the earthquake epicenter and magnitude were used in this study. In the future we hope to apply multifractal techniques to more complete descriptions of the earthquake process (e.g. incorporate the depth and time coordinates, employ the seismic moment tensor, etc.).

The seismic fields were produced on 512 by 512 square grids over a $1000km \times 1000km$ region yielding a minimum resolution of $\approx 2km$. This resolution was chosen so as to be larger than the accuracy of the location measurements while simultaneously frequently containing more than one earthquake per grid box. The latter conditions are necessary since both measurement errors, and the finite number of events in the sample will introduce spurious breaks in the scaling at large λ (small distances). Note that many of the grid boxes contained no events; this is either due to their weak intensity (the minimum detectable amplitude corresponded to magnitude 0), or due to the fact that seismicity – even at extremely low intensity levels — is confined to a fractal subspace with $d < 2$. We discuss this further below.

The basic scaling properties we are interested in are the behavior of the different moments of $S_{\eta,\lambda}$ as the resolution (*i.e.* λ) is varied. In the scaling regime, we define the moment scaling function $K(q,\eta)$ as follows:

$$\langle (S_{\eta,\lambda})^q \rangle \approx \lambda^{K(q,\eta)}, \tag{2}$$

where "$\langle\ \rangle$" indicates statistical (ensemble) averaging. This averaging is necessary since we treat the seismic field as the outcome of a stochastic seismic process. The symbol $\approx$ indicates equality to within constant factors.

We have already mentioned that virtually all the previous scaling results on earthquakes are obtained with $\eta = 0$; for example, the generalized dimension function (Grassberger, 1983; Hentschel and Proccacia, 1983; Schertzer and Lovejoy, 1983) is given by $D_q = D - K(q,0)/(q-1)$ and the box counting and correlation dimensions of seismic events are the special cases D_0, D_2, respectively (Kagan and Knopoff, 1980; Aviles et al., 1987; Hirata et al., 1987; Hirata, 1989; Geilikman et al., 1990). In order to estimate $K(q,\eta)$ for each $S_{\eta,\lambda}$ field we use a generalization (due to ensemble averages) of the partition functions used in literature called the trace moments. This is equivalent to a double trace moment (DTM) analysis (Lavallée, 1991) of the underlying A_Λ field. The trace moment of $S_{\eta,\lambda}$ is defined as:

$$\begin{aligned} Tr\left[(S_{\eta,\lambda})^q\right] &= \left\langle \sum_i \left(S_{\eta,\lambda,i}\lambda^{-D}\right)^q \right\rangle \\ &= \left\langle \sum_i \left(\int_{B_{\lambda,i}} (A_\Lambda)^\eta \, d^D x \right)^q \right\rangle \\ &= \lambda^{K(q,\eta)-(q-1)D}, \end{aligned} \tag{3}$$

where the sum is over all the grid elements at scale λ and indexed by i.

In Fig. 3 one can observe scaling over the entire range of λ, from $2km$ to $1000km$. Figure 4 shows a plot of $K(q,\eta)$ versus q for various values of η. If the field were a monofractal, the lines in Fig. 3 would have slopes which increase as a linear function of q which is not the case, hence seismic fields are multifractal processes. At both the larger and smaller values of q, the $K(q,\eta)$ becomes linear. At smaller values this is due to weak order singularities (*i.e.* smaller earthquakes) either as a result of the detection limits of the seismological network, or

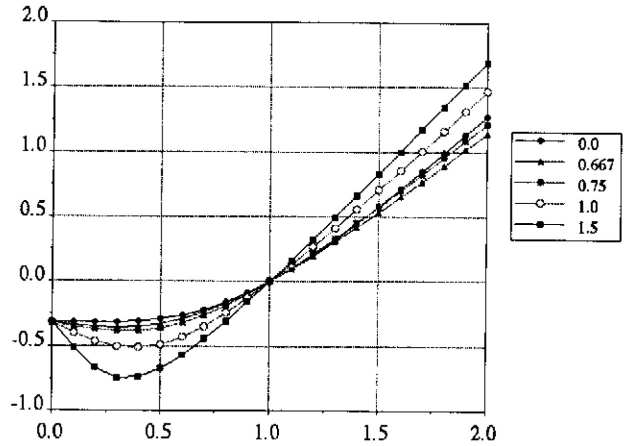

Fig. 4. Statistical Moment Scaling function, $K(q,\eta)$, versus q for various values of η.

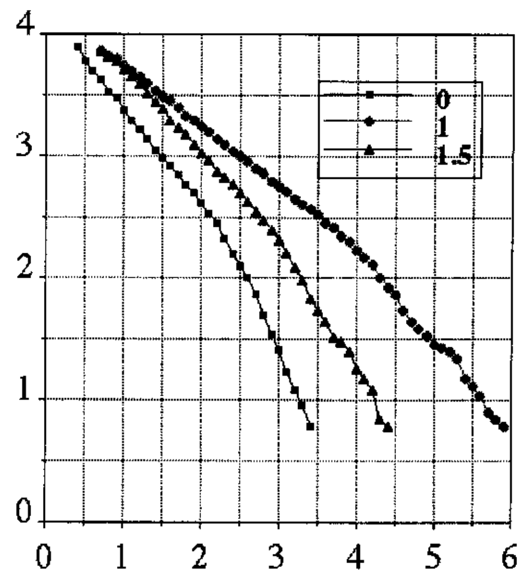

Fig. 5. This figure show the log_{10}(relative frequency of an event) versus $log_{10}(S_{\eta,\lambda})$, using a 512 × 512 grid, for three values of $\eta = 0, 1, 1.5$. The corresponding negative slopes are equal to $1.2 \pm 0.1, 1.0 \pm 0.1, 0.5 \pm 0.1$.

the absence of such events in the underlying multifractal seismic process. The reason $K(q,\eta)$ becomes linear for larger q will be explained below in terms of a first order multifractal phase transition.

4 Generalized Critical Exponents, First Order Multifractal Phase Transitions and Self-Organized Criticality

To generalize the Gutenberg-Richter law to the seismic fields we define the following set of critical exponents $q_{D,\eta}$:

$$Pr(S_{\eta,\lambda} > s) \approx s^{-q_{D,\eta}}, \text{ for } s \gg 1 \tag{4}$$

where Pr indicates "probability", and $q_{D,\eta}$ is the generalized Gutenberg-Richter exponent. This notation anticipates the independence (due to the scaling) of $q_{D,\eta}$ on the resolution λ, but to its nontrivial dependence on the effective dimension D, and the index η. Because of its power law form, the Gutenberg-Richter law is often called "scaling" which is unfortunate since it is only "scaling" with respect to the intensity of the event, whereas the term "scaling" is more properly reserved for power law behaviour under changes in spatial (or temporal) size/resolution. Since the above implies the divergence of high order statistical moments:

$$\langle (S_{\eta,\lambda})^q \rangle \to \infty, \text{ for } q \geq q_{D,\eta} \tag{5}$$

$q_{D,\eta}$ is more properly called the critical exponent of "divergence of moments" and separates two qualitatively different behaviors: the low q *soft* behavior and the high q *hard* behaviour (Schertzer and Lovejoy, 1992). Figure 5 shows the probability histograms for several of the $S_{\eta,\lambda}$ fields defined above. One can see that the probability tail is linear for each value of η ; we estimate $q_{D,\eta}$ from the negative asymptotic slopes.

One of the attractive features of our multifractal model of seismicity is that multifractal processes generically lead, via a specific mechanism called "dressing" described below, to this type of divergence. Since divergence of moments coupled with scaling has been taken as the basic features of "self-organized criticality" (Bak et al., 1987, 1988), Schertzer et al. (1993) and Schertzer and Lovejoy (1994b) have argued that "self-organized criticality" may be a multifractal phenomenon. In any case, no matter what is the origin of the divergence, it will be associated with a qualitative change in the $K(q,\eta)$ function estimated with a finite sample size. This is apparent since empirical values are always finite; for $q \geq q_{D,\eta}$, the empirical estimates of $\langle (S_{\eta,\lambda})^q \rangle$ will depend on sample size and D in a precise way; Schertzer and Lovejoy (1994b) show that empirical $K(q,\eta)$ functions undergo discontinuities in their slopes at $q = q_{D,\eta}$ after which they are linear. The amplitude of the discontinuity is determined by the sample size and D. Figure 4 shows this linear behavior for $q \geq q_{D,\eta}$. Since there is a formal analogy between multifractals and thermodynamics, such qualitative changes are called "multifractal phase transitions", here they are first order (discontinuities in the second derivative can also arise due to sampling effects, even if there is no divergence of moments, see Schertzer and Lovejoy (1994b)).

As a multifractal process proceeds to smaller and smaller scales, it becomes more and more intermittent, being characterized by increasingly violent regions (the singularities) and increasingly calm regions (the regularities). The small scale limit is mathematically singular; in order to obtain well defined limiting properties, it is necessary to integrate (*i.e.* average) the process over

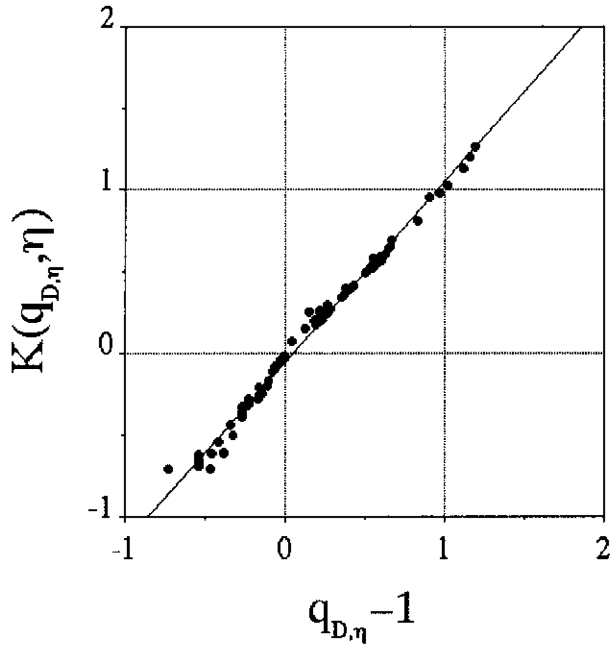

Fig. 6. This figure shows the relation between $K(q_{D,\eta},\eta)$ and $q_{D,\eta}$ for values of η ranging from 0.0 to 2.0 — 100 values of η between. The slope of the best-fit line is 1.1, with an intercept roughly zero. Using earthquake data from the years 1980-90.

finite sets with dimension d. For low order moments, the resulting "dressed" field will have the same scaling properties as the nonintegrated ("bare") process; however, for $q \geq q_{D,\eta}$ the integration fails to sufficiently smooth out the process, one obtains violent "hard" singularities and divergence of the corresponding moments. The exact order is given by the solution of the following equation:

$$K(q_{D,\eta},\eta) = (q_{D,\eta} - 1)D \tag{6}$$

which is a consequence of applying the formula for $\eta = 1$ to η powers of the underlying (bare) process (Schertzer and Lovejoy, 1987, 1994a). It should be emphasized that this equation is a theoretical prediction of the theory of general ("canonical") multifractals and applies only when the source of the divergence is this dressing (smoothing/averaging) mechanism. It is therefore of interest to test this equation so as to discover whether the observed $K(q,\eta)$ and $q_{D,\eta}$ can be explained this simple way. Due to the difficulty of measuring very weak but frequent seismic events, it is not immediately obvious whether or not seismic processes generate such events. If they do not, and the process is confined to a fractal subspace, then the dimension D in Eq.(6) will be less than d. Here we rather regard D as an empirically determined parameter, which we estimate directly by plotting $K(q_{D,\eta},\eta)$ against $q_{D,\eta}$.

Figure 6 shows this relation for values of η ranging from 0.0 to 2.0 and using earthquake data from the years 1980-90. The slope of the line is 1.1. This confirms the relation with a dressing dimension of $D \approx 1.1$.

5 Conclusions

Until now, the two basic empirical laws about earthquakes, the spatial scaling of their distribution (the hypocenters form a fractal set, the density, a multifractal measure), and the divergence of statistical moments (the Gutenberg-Richter law) have not simultaneously coexisted in a coherent theoretical framework. Even deterministic models exhibiting self-organized criticality fail to provide a general connection between the two. Largely as a consequence of this, empirical analyses have generally not been able to simultaneously deal with the spatial distribution of the earthquakes and with their intensities. We have argued here that the fundamental seismic processes are scaling space-time tensor (e.g. stress-strain) processes involving (tensor) space-time multifractal fields resulting from Lie cascades. Although the observed ground displacements (and the associated seismic fields) are non-trivially (and nonlinearly) related to these processes, we will nevertheless expect multifractals to provide the appropriate theoretical framework and analysis methods. This motivates the study of the (normalized) powers of seismic fields from the USGS earthquake catalogue by summing various powers of ground displacements onto grids. With only one exception (η=1.5, Hirabayashi et al. (1992)) existing scaling analysis has been on the special case $\eta = 0$ — the only case with no intensity information. By applying a multifractal analysis technique (trace moments) on all the members, we show that the seismic fields exhibit characteristics typical of multifractals. Finally, using multifractal theory, we show that multiscaling of the seismic fields leads via multifractal phase transitions to (generalized) Gutenberg-Richter exponents $q_{D,\eta}$. An important consequence is that multifractality, although theoretically present for any η and q is only *directly* observable for $q \leq q_{D,\eta}$. These exponents are shown to obey a simple theoretically predicted formula which arises due to the "dressing" of the fundamental seismic fields. Contrary to the usual deterministic framework which situates the origin of the self-organized critical behaviour of earthquakes in deterministic toy-models, we demonstrated the possibility of an alternative: self-organized criticality of earthquakes can originate from stochastic space-time tensorial multifractal processes. We also pointed out the necessity to proceed to multifractal tensorial analysis with the help of Lie analysis to better taking into account many features of the seismicity which are beyond the present scalar multifractal analysis.

Acknowledgements. The authors would like to thank O. Jensen and Y. Kagan for many discussions and criticisms of earlier drafts of this paper. Also, we would like to thank D. Lavallée, J. Todoeschuck, and H. Beltrami for their comments and suggestions.

D. Oppenheimer from the USGS Menlo Park, California is thanked for his assistance in obtaining the earthquake data as well as answering numerous questions.

References

Aviles, C. A., C. H. Scholz, and J. Boatwright, Fractal Analysis Applied to Characteristic Segments of the San Andreas Fault, *J. Geophys. Res., 92*, 331, 1987.

Bak, P., C. Tang, and K. Weisenfeld, Self-Organized Criticality: An Explanation of 1/f Noise, *Physical Review Letters, 59*, 381, 1987.

Bak, P., C. Tang, and K. Weisenfeld, Self-Organized Criticality, *Physical Review Letters A, 38*, 364, 1988.

Barriere, B. and D. Turcotte, A Scale Invariant Cellular Automata Model for Distributed Seismicity, *Geophys. Res. Letters, 18*, 11, 1991.

Geilikman, M.B., T. V. Golubeva, and V. F. Pisarenko, Multifractal Patterns of Seismicity, *Earth and Planetary Science Letter, 99*, 127, 1990.

Grassberger, P., Generalized Dimensions of Strange Attractors, *Phys. Lett., A97*, 1983.

Gutenberg, B. and C. F. Richter, Frequency of Earthquakes in California, *Bull. Seis. Soc. Amer., 34*, 185, 1944.

Hentschel, H.G.E., and I. Proccacia, The Infinite Number of Generalized Dimensions of Fractals and Strange Attractors, *Physica 8D*, 435, 1983.

Hirabayashi, T., K. Ito, T. Yoshii, Multifractal analysis of earthquakes, *Pageoph, 138*, 591, 1992.

Hirata, T. and M. Imoto, Multi-fractal Analysis of Spatial Distribution of Microearthquakes in the Kanto Region, *Geophys. J. Intl., 107*, 155, 1991.

Hirata, T., T. Satoh, and K. Ito, Fractal Structure of Spatial Distribution of Microfracturing in Rock, *Geophys. J. R. Astr. Soc., 90*, 369, 1987.

Hirata, T., Fractal dimension of fault systems in Japan: Fractal structure in rock fracture geometry at various scales, in *Fractals in Geophysics*, edited by C. Scholz and B. B. Mandelbrot, pages 157–70, Birkhauser, Boston, 1989.

Hooge, C., *Earthquakes as a Space-Time Multifractal Process*, M.Sc. thesis, McGill University, Montréal, 1993.

Ito, K. and M. Matsuzaki, Earthquakes as Self-Organized Critical Phenomenon, *J. Geophys. Res., 95*, 1990.

Kagan Y., and L. Knopoff, The Spatial Distribution of Earthquakes: The two point correlation function, *Geophys. J. R. Astron. Soc., 62*, 303, 1980.

Lavallée, D., *Multifractal Analysis and Simulation Techniques and Turbulent Fields*, PhD thesis, McGill University, Montréal, 1991.

Marks, S.M. and F. W. Lester, *Catalog of earthquakes along the San Andreas faults system in Central California, April-June 1977*, Open-file report 80-1264, U.S. Geological Survey, Menlo Park, California, 43 pp., 1980.

Okubo, P.G. and K. Aki, Fractal Geometry in the San Andreas Fault System, *J. Geophys. Res., 92*, 345, 1987.

Omori, F., On the After-Shocks of Earthquakes, *J. Coll. Sci., 7*, 111, 1895.

Sadovskiy, M.A., T. V. Golubeva, V. F. Pisarenko, and M. G. Shnirman, Characteristic Dimensions of Rock and Hierarchical Properties of Seismicity, *Izv. Acad. Sci. USSR Phys. Solid Earth, 20*, 87, 1984.

Schertzer, D. and S. Lovejoy, The dimension of atmospheric motions, in *IUTAM Symp. on Turbulence and Chaotic Phenomena in Fluids*, pages 141–144, Kyoto, Japan, 1983.

Schertzer, D. and S. Lovejoy, The dimension and intermittency of atmospheric dynamics, *Turbulent Shear flow 4*, 7-33 ed. B. Launder, Springer, 1985.

Schertzer, D. and S. Lovejoy, Physical Modeling and Analysis of Rain and Clouds by Anisotropic Scaling Multiplicative Processes, *J. Geophys. Res., 92*, 9693, 1987.

Schertzer, D. and S. Lovejoy, *Non-Linear Variability in Geophysics: Scaling and Fractals*, eds. D. Schertzer and S. Lovejoy, Kluwer, Norwell, MA, 318 pp., 1991.

Schertzer, D. and S. Lovejoy, Hard and Soft Multifractal Processes *Physica A, 185*, 187, 1992.

Schertzer, D. and S. Lovejoy, From scalar cascades to Lie cascades: joint multifractal analysis of rain and cloud processes, *Space/time Variability and Interdependance for Various Hydrological Processes*, Ed. R.A. Feddes, Cambridge University Press, 1994a (in press).

Schertzer, D., S. Lovejoy and D. Lavallée, Generic Multifractal Phase Transitions and Self-Organized Crtiticality, in *Cellular Automata: Prospects in Astrophysical Applications* eds. A Lejeune and J. Perdang, World Scientific, Singapore, 1993.

Schertzer, D. and S. Lovejoy, Multifractal generation of self-organized criticality, in *Fractals in Natural and Applied Science*, ed. M. M. Novak, IFIP, p. 325-339, 1994b.

Tessier, Y., S. Lovejoy, D. Schertzer, The multifractal global raingage network: analysis and simulations, *J. Appl. Meteor.*, 1994 (in press).